Colored Spin Systems, BKP Evolution and finite N_c effects

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Abstract

Even within the framework of the leading logarithmic approximation the eigenvalues of the BKP kernel for states of more than three reggeized gluons are unknown in general, contrary to the planar limit case where the problem becomes integrable. We consider a 4-gluon kernel for a finite number of colors and define some simple toy models for the configuration space dynamics, which are directly solvable with group theoretical methods. Then we study the dependence of the spectrum of these models with respect to the number of colors and make comparisons with the large limit case.

1 Introduction

In quantum field theory and statistical mechanics the $1/N$ (or large N) expansion [1] is a well known and extensively used perturbative framework whenever the theories under investigation present an internal symmetry typically related to groups like $SO(N)$ or $SU(N)$.

Quantum Chromodynamics is one of the theories mostly studied under this approximation even if, as a physical gauge theory, it is characterized by a gauge group $SU(N_c)$ where the number of colors N_c is just 3. Recently, thanks to the renewed interest induced by the ADS/CFT correspondence, the $N = 4$ SYM theory in the infinity color (planar) limit has been intensively studied and several important results achieved.

The fact that the planar $N = 4$ SYM is expected by the theoretical community to be solvable and that it is dual to a superstring sigma model has led several theorists to look for hints, in the absence of any supersymmetry, for the existence of a possible dynamical system dual to planar QCD sharing with it some integrability properties. The starting points are the

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integrable structures unveiled many years ago at one loop in standard perturbation theory and some hints of possible integrability at two loops in the planar limit.

The first evidence of an integrable structure at one loop in QCD was found [2] by L.N. Lipatov in the framework of the Regge limit of scattering amplitudes whose behavior may be conveniently described by systems of interacting reggeized gluons, as we shall briefly review in the next section. The integrable dynamics, associated to the evolution in rapidity of such a system, appears when one is taking the large N_c approximation, which makes the BKP kernel [3] to resemble the structure of an Heisenberg XXX spin chain, but for a non compact $SL(2,\mathbb{C})$ "spin".

Going beyond the large N_c approximation, even in the lowest orders in perturbation theory in the coupling constant, is a formidable task and it is very difficult also to try to estimate the error one faces when computing quantities for infinite N_c (planar limit) instead of at $N_c = 3$.

It is the purpose of this work to introduce some finite toy models, which share the same color structure of the BKP systems and can be studied to determine the dependence of the spectrum on the number of colors N_c. They are characterized by a configuration space which is no more the transverse plane but a finite vector space associated to irreducible representations of the $SU(2)$ group so that one may use group theoretical methods to analyse some of these models.

This is of course not providing any concrete answer for the question related to the real QCD problem, but nevertheless can be of some help. Moreover some toy models may be interesting by themselves as dynamical systems.

We start in the next section with a short review of the properties of the system of interacting reggeized gluons in the Leading Logarithmic Approximation. In section three we consider the color structure for the four reggeized gluon system and describe how to use a convenient basis for it. In section four we construct some finite toy models which are studied in some details in a couple of subsections. After the conclusions in few appendices we give more details on symmetry structures and on the features of these toy models.

2 BKP Kernels

Let us start by giving a brief overview of the kernels which encode the evolution in rapidity of systems of interacting reggeized gluons in the leading logarithmic approximation (LLA). The reggeized gluons provide a convenient perturbative description of part of the QCD degrees of freedom in the Regge limit (also known as the small x limit) and appeared in the investigations of the leading dependence of the total cross sections on the center of mass energy in the LLA, which is associated to the so called BFKL (perturbative) pomeron [4]. In the simplest form, the BFKL pomeron turns out to be a composite state of two interacting reggeized gluons "living" in the transverse configuration plane in the colorless configuration. The construction of the kernel reflects the property that in the Regge limit the scattering amplitude factorizes in the impact factors which determine the coupling of the external particles to the t-channel reggeized gluons and in a Green's function which exponentiates the

kernel and contains the rapidity dependence of their composite state. Such a dependence can be analyzed in terms of the spectral properties of the kernel and in particular one is interested in the eigenvalues and eigenstates associated to the leading behavior. Because of this the spectral problem is often formulated in quantum mechanical terms with the kernel being the "Hamiltonian" and its eigenvalues the "energies".

In the case of a colorless exchange the Hamiltonian is infrared finite and in LLA is constructed summing the perturbative contributions of different Feynman diagrams: in particular the virtual ones (reggeized gluon trajectories) ω and the real ones (associated to an effective real gluon emission vertex) V. One writes formally $H = \omega_1 + \omega_2 + \vec{T}_1 \vec{T}_2 V_{12}$ where $\vec{T}_i$ are the generators of the color group in adjoint representation . In the colorless case one has $\vec{T}_1 \vec{T}_2 = -N_c$ and finally one obtains:

$$H_{12} = \ln |p_1|^2 + \ln |p_2|^2 + \frac{1}{p_1 p_2^*} \ln |\rho_{12}|^2 \, p_1 p_2^* + \frac{1}{p_1^* p_2} \ln |\rho_{12}|^2 \, p_1^* p_2 - 4\Psi(1) \, , \tag{2.1}$$

where $\Psi(x) = d \ln \Gamma(x)/dx$, a factor $\bar{\alpha}_s = \alpha_s N_c / \pi$ has been omitted and the gluon holomorphic momenta and coordinates have been introduced.

The gauge invariance gives the freedom to choose a description within the Möbius space [5, 6], wherein the functions describing the positions of the two reggeized gluons in the transverse plane are zero in the coincidence limit. In this space the BFKL hamiltonian has the property of the holomorphic separability ($H_{12} = h_{12} + \bar{h}_{12}$). Moreover a remarkable property is its invariance under the Möbius group, whose generators for the holomorphic sector in the Möbius space for the principal series of unitary representations are given by:

$$M_r^3 = \rho_r \partial_r \, , \; M_r^+ = \partial_r \, , \; M_r^- = -\rho_r^2 \partial_r \, . \tag{2.2}$$

The associated Casimir operator for two gluons is

$$M^2 = |\vec{M}|^2 = -\rho_{12}^2 \, \partial_1 \, \partial_2 \, , \tag{2.3}$$

where $\vec{M} = \sum_{r=1}^2 \vec{M}_r$ and $\vec{M}_r \equiv (M_r^+, M_r^-, M_r^3)$. Due to this symmetry the holomorphic and antiholomorphic parts of the Hamiltonian can be written explicitly in terms of the Casimir operator: indeed one has, after defining formally $J(J-1) = M^2$,

$$h_{12} = \psi(J) + \psi(1-J) - 2\psi(1) \, . \tag{2.4}$$

Labelled by the conformal weights $h = \frac{1+n}{2} + i\nu$, $\bar{h} = \frac{1-n}{2} + i\nu$, where n is the conformal spin and $d = 1 - 2i\nu$ is the anomalous dimension of the operator $O_{h,\bar{h}}(\boldsymbol{\rho}_0)$ describing the compound state [7], the eigenstates and eigenvalues of the full hamiltonian in eq. (2.1), $H_{12} E_{h,\bar{h}} = 2\chi_h E_{h,\bar{h}}$, are respectively given by:

$$E_{h,\bar{h}}(\boldsymbol{\rho}_{10}, \, \boldsymbol{\rho}_{20}) \equiv \langle \rho | h \rangle = \left(\frac{\rho_{12}}{\rho_{10}\rho_{20}} \right)^h \left(\frac{\rho_{12}^*}{\rho_{10}^* \rho_{20}^*} \right)^{\bar{h}} \, , \tag{2.5}$$

and

$$\chi_h \equiv \chi(\nu, n) = \psi\left(\frac{1+|n|}{2} + i\nu\right) + \psi\left(\frac{1+|n|}{2} - i\nu\right) - 2\psi(1)\,. \tag{2.6}$$

The leading eigenvalue, at the point $n = \nu = 0$, has a value $\chi_{max} = 4\ln 2 \approx 2.77259$, responsible for the rise of the total cross section as $s^{\bar{\alpha}_s \chi_{max}}$, which corresponds to a strong violation of unitarity.

Let us now consider the evolution in rapidity of composite states of more than 2 reggeized gluons [3]. The BKP Hamiltonian in LLA acting on a colorless state can be written in terms of the BFKL pomeron Hamiltonian and has the form (see [2])

$$H_n = -\frac{1}{N_c} \sum_{1\leq k<l\leq n} \vec{T}_k \vec{T}_l H_{kl}\,. \tag{2.7}$$

This Hamiltonian is conformal invariant but cannot be solved in general. Nevertheless the case of three reggeized gluons, where the color structure trivially factorizes, is solvable [2] and different families of solutions were found [8, 9]. Physically these states are associated to the so called odderon exchange [10] and in particular the family of solutions given in [9] corresponds to eigenvalues up to zero (intercept up to one) and are the leading one in the high energy limit. Moreover they have a non null coupling to photon-meson impact factors [11].

The case of more than three reggeized gluon is in general not solvable but if one considers the color cylindrical topology when taking the large N_c limit the resulting Hamiltonian

$$H_n^\infty = \frac{1}{2}\left[H_{12} + H_{23} + \cdots + H_{n1}\right] = h_n + \bar{h}_n \tag{2.8}$$

is integrable, i.e. there exists a set of other $n-1$ operators q_r, which commute with it and are in involution. They are given, in coordinate representation, by

$$q_r = \sum_{i_1<i_2<\cdots<i_r} \rho_{i_1 i_2}\rho_{i_2 i_3}\cdots\rho_{i_r i_1}\, p_{i_1} p_{i_2}\cdots p_{i_r}\,, \tag{2.9}$$

together with similar relations for the antiholomorphic sector. In particular, $q_2 = M^2$ is the Casimir of the Möbius group. This is the first case where integrability was found within the context of gauge theories analyzing the Green's function in some kinematical limit. This integrable model is a non compact generalization of the Heisenberg XXX spin chains [2, 12] and has been intensively studied with different techniques in the last decade [13, 14, 15, 16, 17, 18].

Here we remind the result for the highest eigenvalue of a system of four reggeized gluons in the planar, one cylinder topology (1CT), case: $H_4^\infty \psi_4 = 2E_4^{\rm 1CT}\psi_4$. The maximum value found, for zero conformal spin, is

$$E_4^{\rm 1CT} = 0.67416\,. \tag{2.10}$$

In general for an arbitrary number n of reggeized gluon in the cylindrical topology the leading eigenvalues have been found to be positive for even n and negative for odd n and asymptotically behaving as $1/n$ [15, 16].

The following are two important question that are unfortunately very hard to answer: what are the eigenvalues at finite $N_c = 3$ and what is in general their dependence in N_c? One may be tempted to apply variational or perturbative techniques to the spectral problem, which nevertheless appears to be quite involved. In any case a first step consists of analyzing the color structure, which simplifies a bit in the case of four reggeized gluons in a total colorless state.

3 Color structure

We consider the BKP kernel H_4 for four gluons, given in eq. (2.7). This is an operator acting on 4-gluon states, which may be represented as functions of the transverse plane coordinates and of the gluon colors $v(\{\boldsymbol{\rho}_i\})^{a_1a_2a_3a_4}$. Let us concentrate here on the color space.

It is convenient, due to the fact that the four gluons are in a total color singlet state, to write the color vector $v^{a_1a_2a_3a_4}$ in terms of the color state of a two gluon subchannel. Let us therefore start from the resolution of unity for a state of two $SU(N_c)$ particles in terms of the projectors $P[R_i]_{a_1a_2}^{a_1'a_2'}$ onto irreducible representations:

$$1 = P_1 + P_{8A} + P_{8S} + P_{10+\bar{10}} + P_{27} + P_0 = \sum_i P[R_i]\,, \tag{3.11}$$

where $TrP[R_i] = d_i$ is the dimension of the corresponding representation. Let us note that we have chosen to consider a unique subspace for the direct sum of the two spaces corresponding to 10 and $\bar{10}$ representations. This is convenient for our purposes and we shall therefore consider just 6 different projectors to span the color space of two gluons.

If we consider gluons $(1, 2)$ to be the reference channel we introduce as the base for the color vector space the set $\{P[R_i]_{a_1a_2}^{a_3a_4}\}$ of projectors and write

$$v^{a_1a_2a_3a_4} = \sum_i v^i \left(P[R_i]_{a_1a_2}^{a_3a_4}\right) \quad \text{or} \quad v = \sum_i v^i P_{12}[R_i]\,. \tag{3.12}$$

Note that one could have also chosen other reference channels corresponding to a description in terms of projection onto irreducible representations of other gluon subsystems. Having chosen a color basis, we find that the next step is to write the BKP kernel with respect to it. We can slightly simplify the expression for the kernel since for a colorless state we have $\sum_i \vec{T}_i v = 0$ which implies that $\vec{T}_1\vec{T}_2 v = \vec{T}_3\vec{T}_4 v$ (an similarly for the other permutations of the indices). Therefore one may write:

$$H_4 = -\frac{1}{N_c}\left[\vec{T}_1\vec{T}_2\,(H_{12} + H_{34}) + \vec{T}_1\vec{T}_3\,(H_{13} + H_{24}) + \vec{T}_1\vec{T}_4\,(H_{14} + H_{23})\right]. \tag{3.13}$$

Let us now write explicitly the action of the color operators $\vec{T}_i\vec{T}_j = \sum_a T_i^a T_j^a$ which are associated to the interaction between the gluons labelled i and j. We start from the simple

"diagonal channel" for which we have relation $\vec{T}_i\vec{T}_j = -\sum_k a_k P_{ij}[R_k]$ with coefficients $a_k = (N_c, \frac{N_c}{2}, \frac{N_c}{2}, 0, -1, 1)$. Consequently we can write in the $(1,2)$ reference base

$$\left(\vec{T}_1\vec{T}_2 v\right)^j = -a_j v^j = -\left(A\, v\right)^j \;, \tag{3.14}$$

where $A = diag(a_k)$. The action on v of the $\vec{T}_1\vec{T}_3$ and $\vec{T}_1\vec{T}_4$ operators is less trivial and is constructed in terms of the $6j$ symbols of the adjoint representation of $SU(N_c)$ group. We shall give few details in the appendix A and write directly the results, in terms of the symmetric (after a similarity transformation) matrix operators:

$$\left(\vec{T}_1\vec{T}_3\, v\right)^j = -\sum_i \left(\sum_k C^j_k a_k C^k_i\right) v^i = -\left(C A\, C\, v\right)^j \tag{3.15}$$

and

$$\left(\vec{T}_1\vec{T}_4\, v\right)^j = -\sum_i \left(\sum_k s_j C^j_k a_k C^k_i s_i\right) v^i = -\left(SCA\, CS\, v\right)^j \;. \tag{3.16}$$

The matrix C is the crossing matrix built on the $6j$ symbols and $S = diag(s_j)$ is constructed on the parities $s_j = \pm 1$ of the different representations R_j.

We can therefore write the general BKP kernel for a four gluon state, given in eq. (3.13), as

$$H_4 = \frac{1}{N_c}\left[A\left(H_{12} + H_{34}\right) + CAC\left(H_{13} + H_{24}\right) + SCACS\left(H_{14} + H_{23}\right)\right] \tag{3.17}$$

One can check that if we make trivial the transverse space dynamics, replacing the H_{ij} operators by a unit operators, the general BKP kernel in eq. (2.7) becomes $H_n = \frac{n}{2}\hat{1}$ and indeed one can verify that $A + CAC + SCACS = N_c\hat{1}$.

Let us make few considerations on the large N_c limit approximation. As we have already discussed, in the Regge limit one faces the factorization of an amplitude in impact factors and a Green's function which exponentiates the kernel. The topologies resulting from the large N_c limit depend on the impact factor structure. In particular one expects the realization of two cases: the one and two cylinder topologies. The former corresponds to the case, well studied, of the integrable kernel, Heisenberg XXX spin chain-like. It is encoded in the relation: $\vec{T}_i\vec{T}_j \to -\frac{N_c}{2}\delta_{i+1,j}$ which leads to $H_4 = \frac{1}{2}\left(H_{12} + H_{23} + H_{34} + H_{41}\right)$. It is characterized by eigenvalues corresponding to an intercept less then a pomeron. The latter case instead is expected to have a leading intercept, corresponding to an energy dependence given by two pomeron exchange. Consequently one expects at finite N_c a contribution with an energy dependence even stronger. In the two cylinder topology the color structure is associated to two singlets ($\delta_{a_1a_2}\delta_{a_3a_4}$, together with the other two possible permutations). Such a structure is indeed present in the analysis, within the framework of extended generalized LLA, of unitarity corrections to the BFKL pomeron exchange [19] and diffractive dissociation in DIS [20], where the perturbative triple pomeron vertex (see also [21]) was discovered and shown to couple exactly to the four gluon BKP kernel.

It is therefore of great importance to understand how much the picture derived in the planar $N_c = \infty$ case is far from the real situation with $N_c = 3$. One clearly expects for example that the first corrections to the eigenvalues of the BKP kernel are proportional to $1/N_c^2$, but what is unknown is the multiplicative coefficient as well as the higher order terms.

4 Toy models

In this section we shall consider a family of models, different from the BKP system, which neverthelss share several features with it and can be used to judge how the large N_c approximation might be more or less satisfactory. Moreover these systems may be considered interesting by themselves as quantum dynamical systems.

A state of n reggeized gluons undergoing the BKP evolution, described by the kernel in eq. (2.7), belongs to a vector space of functions on a domain given by the tensor product of the color space $\mathbf{8^n}$ and the configuration space $\mathbb{R}^{2n}$, associated to the position or momenta in the transverse plane, of the n gluons. Indeed the BKP kernel is built as a sum of product of color $(\vec{T}_k \vec{T}_l)$ and of configuration (H_{kl}) operators; the latter, on the Möbius space, can be written in terms of the Casimir of the Möbius group, i.e. in terms of the scalar product of the generators of the non compact spin group $SL(2, \mathbb{C})$: $H_{kl} = H_{kl}(\vec{M}_k \cdot \vec{M}_l)$.

We are, therefore, led to consider a class of toy models where the BKP configuration space $\mathbb{R}^{2n}$ is substituted by the space V_s^n where V_s is the finite space spanned by spin states belonging to the irreducible representation of $SU(2)$ with spin s. In particular we shall consider quantum systems with an Hamiltonian fitting the following structure:

$$\mathcal{H}_n = -\frac{1}{N_c} \sum_{1 \le k < l \le n} \vec{T}_k \vec{T}_l \, f(\vec{S}_k \vec{S}_l) \,, \tag{4.18}$$

where $\vec{S}_i$ are the elements of the $su(2)$ algebra associated to the particle i in any chosen representation and f is a generic function. A particular toy model is therefore specified by giving the spin s of each particle ("gluons") and the function f. In the following we shall consider two specific cases for the 4 particle system:

a) A spin $s = 1$ case in a global singlet state v $(\sum_i \vec{S}_i \, v = 0)$. If f is the identity map than the "spin" configuration dynamics is very similar to the one of the color sector. In order to have a system which behaves similarly to the BKP case we first put a constraint on the two particle operators, which describe the basic interaction. In particular we consider the family of functions

$$f_\alpha(x) = 2\mathrm{Re}\left[\psi\left(\frac{1}{2} + \sqrt{-\alpha(4+2x)}\right)\right] - 2\psi(1) \,. \tag{4.19}$$

Remembering that for conformal spin $n = 0$ the BFKL Hamiltonian is given by $H_{kl} = 2\mathrm{Re}\left[\psi\left(\frac{1}{2} + \sqrt{\frac{1}{4} + (\vec{M}_k + \vec{M}_l)^2}\right)\right] - 2\psi(1)$, one immediately recognizes that the f_α is associated to the substitution $\frac{1}{4} + L_{ij}^2 \to -\alpha \, S_{ij}^2$ which assures to have the same leading eigenvalue

for any α, since both expressions have the value zero as upper bound. The parameter α will be chosen in order to constrain the full 4-particle Hamiltonian (4.18) to have the same leading eigenvalue as the QCD BKP system in the large N_c limit (at zero conformal spin). In this system, the BKP toy model, we shall investigate finite N_c effects.

b) A system $\mathrm{TOY}_{\mathrm{Adj,Fund}}$ with f the identity function and spin $s = 1/2$. Such a system in the large N_c limit in the case of one cylinder topology becomes the well known Heisenberg XXX spin chain system which is integrable. We shall perform some check on the N_c dependence again for the 4-particle case.

c) Moreover in order to have another check of the approach we shall also consider a model where the 4 particle belong to the fundamental representation of $SU(2)$ for both the "color" and the "spin" so that we can perform a comparison with standard results from the spectroscopy of isospin-spin systems. We place these checks in the appendix C.

4.1 BKP toy model

In order to explicitly study this finite system, described by the Hamiltonian in eq. (4.18) acting on vector states with dimension $(8\times3)^4$ and singlet under both $SU(3)_C$ and $SU(2)_{spin\ conf}$, it is convenient to choose the color decomposition in 2-particle subchannel irreducible representations described in section 3 and adopt a similar approach also on the "spin" degrees of freedom. After that one is left with the problem of diagonalizing an Hamiltonian which is a matrix 18×18, a problem addressable with any computer. Without the singlet restriction on the spin part the problem in general is much more complicated to be easily solved and may be addressed in future investigations.

Let us therefore proceed by introducing for 2 particle spin 1 states the resolution of unity $1 = Q_1 + Q_3 + Q_5 = \sum_i Q[R_i]$ which let us write $\vec{S}_i\vec{S}_j = -\sum_k b_k Q_{ij}[R_k]$ with $b_k = (2, 1, -1)$ (c.f. with a_k: first, second and second last terms). It is, therefore, straightforward to write from a power series representation ($Q_{ij}[R_k]$ are projectors):

$$f(\vec{S}_i\vec{S}_j) = \sum_k f(-b_k) Q_{ij}[R_k]\,. \tag{4.20}$$

Using the corresponding crossing matrices D and the parity matrix S' one obtains relations very similar to the one reported in eqs. (3.14)-(3.16), which read

$$\left(f\left(\vec{S}_1\vec{S}_2\right)v\right)^j = f(-b_j)v^j = (B\,v)^j\ , \tag{4.21}$$

$$\left(f\left(\vec{T}_1\vec{T}_3\right)\ v\right)^j = \sum_i \left(\sum_k D^j_k f(-b_k) D^k_i\right) v^i = (DB\,D\,v)^j \tag{4.22}$$

and

$$\left(f\left(\vec{T}_1\vec{T}_4\right)\ v\right)^j = \sum_i \left(\sum_k s'_j D^j_k f(-b_k) D^k_i s'_i\right) v^i = (S'DB\,DS'\,v)^j\ . \tag{4.23}$$

From the above results for the two particle representation basis, we can write the explicit form of the Hamiltonian for this toy model, going beyond the one given in eq. (3.17). Indeed we obtain

$$\mathcal{H}_{4a} = \frac{2}{N_c} \left(A \otimes B + C A\, C \otimes DB\, D + SC A\, CS \otimes S' DB\, DS' \right) \tag{4.24}$$

which contains a dependence on N_c and on the parameter α through the function f_α given in eq. (4.19).

Let us note that in the large N_c limit one faces for the Hamiltonian two possible cases: the one cylinder topology (1CT) which corresponds to the simpler Hamiltonian

$$\mathcal{H}_{4a}^{1CT} = -\frac{1}{N_c} \left[-\frac{N_c}{2} \sum_i f\left(\vec{S}_i \vec{S}_{i+1} \right) \right] = B + S' DBDS' \tag{4.25}$$

and the two cylinder topology (2CT) corresponding to the even simpler Hamiltonian

$$\mathcal{H}_{4a}^{2CT} = -\frac{1}{N_c} \left[-N_c f\left(\vec{S}_1 \vec{S}_2 \right) - N_c f\left(\vec{S}_3 \vec{S}_4 \right) \right] = 2B\,. \tag{4.26}$$

Let us remark that while in the case of $N_c > 3$ we consider a basis for the vector states made of $P[R_i]Q[R_j]$ with 18 elements since in the color sector there is also the P_0 projector, the case $N_c = 3$ is characterized by a basis of 15 elements.

As already anticipated, in order to study a toy model resembling the spectrum of the BKP system of 4 gluons, we require that, in the large N_c limit in the one cylinder topology, the leading eigenvalue must be the same as the one found for the corresponding integrable BKP system, whose value was given in eq. (2.10). This fact fixes the value of the parameter $\alpha = 2.80665$. We are therefore left with an Hamiltonian which is just a function of the number of colors N_c.

Let us now consider its spectrum for the cases $N_c = 3$ and $N_c = \infty$. Here we report the values followed by their multiplicities. Note than for $N_c = 3$ there are 15 eigenvalues while they are 18 for any other value of N_c. For the case $N_C = \infty$ we specify also the topology they belong to.

$$\begin{pmatrix} N_c = 3 & \\ \mathbf{7.04193} & (\times 1) \\ \mathbf{5.51899} & (\times 2) \\ \underline{1.12269} & (\times 2) \\ -3.89328 & (\times 2) \\ -4.04744 & (\times 1) \\ -4.27838 & (\times 1) \\ -7.81242 & (\times 1) \\ -9.18576 & (\times 2) \\ -12.6743 & (\times 2) \\ -14.1005 & (\times 1) \end{pmatrix} \rightarrow \begin{pmatrix} N_c = \infty & \\ \mathbf{5.54518} & (\times 3)\ 2CT \\ \underline{0.67416} & (\times 3)\ 1CT \\ -4.27838 & (\times 3)\ 1CT \\ -7.81242 & (\times 3)\ 2CT \\ -8.67983 & (\times 3)\ 1CT \\ -10.0168 & (\times 3)\ 2CT \end{pmatrix}$$

We track the flow from $N_C = 3$ to $N_c = \infty$: the first three highest eigenvalues (in bold) are moving to the same leading value (in bold) which corresponds to two BFKL pomeron exchange (in two cylinder topology). The fourth and fifth highest eigenvalues (underlined) are instead moving to the leading eigenvalues of the one cylinder topology case (which are three instead of two because of the larger basis for $N_c > 3$). With very good approximation one finds that the N_c dependence of the leading eigenvalue E_0 is given by

$$E_0(N_c) = E_0(\infty)\left(1 + \frac{2.465}{N_c^2}\right) . \tag{4.27}$$

One can see that for this toy model the large N_c approximation corresponds to an error of about 27%, an error which is not negligible because the coefficient of the leading correction to the asymptotic value, proportional to $1/N_c^2$, is a large number.

It is also easy to investigate the color-configuration space mixing which is encoded in the eigenvectors. We report some results in the appendix B.

4.2 $\mathrm{TOY}_{\mathrm{Adj,Fund}}$

We now move to study the toy model described at point (b) at the beginning of section 4, again to see how the large N_c approximation works. It is described by the Hamiltonian

$$\mathcal{H}_{Adj,Fund} = -\frac{1}{N_c} \sum_{1 \le k < l \le n} \vec{T}_k \vec{T}_l \frac{\vec{\sigma}_k}{2} \frac{\vec{\sigma}_l}{2} , \tag{4.28}$$

acting on spin singlet states. Again we consider the large N_c limit. The one cylinder topology is associated to the well known Heisenberg XXX spin chain with Hamiltonian

$$\mathcal{H}^{1CT}_{Adj,Fund} = \frac{1}{2} \sum_{i=1}^{n} \frac{\vec{\sigma}_i}{2} \frac{\vec{\sigma}_{i+1}}{2} , \tag{4.29}$$

which we shall now consider for the case of $n = 4$ particle. In this case at large N_c we have, as before, also the two cylinder topology associated to the Hamiltonian

$$\mathcal{H}^{2CT}_{Adj,Fund} = 2 \frac{\vec{\sigma}_1}{2} \frac{\vec{\sigma}_2}{2} . \tag{4.30}$$

The spectrum for the one cylinder topology case is well known from Bethe Ansatz methods [22] and for total zero spin of a 4-particle spin chain the possible eigenvalues are 0 and -1 (see table II in [23] for $J = -1/2$ in their notation). The two cylinder topology is characterized by the eigenvalues $+1/2$ and $-3/2$.

At finite N_c we rewrite the Hamiltonian in a similar way to the BKP toy model case (see eq. (4.24) where the B and D matrices are defined for f the identity map and for the group $SU(2)$ in fundamental representation). At $N_c = 3$ it corresponds to a 10×10 matrix while

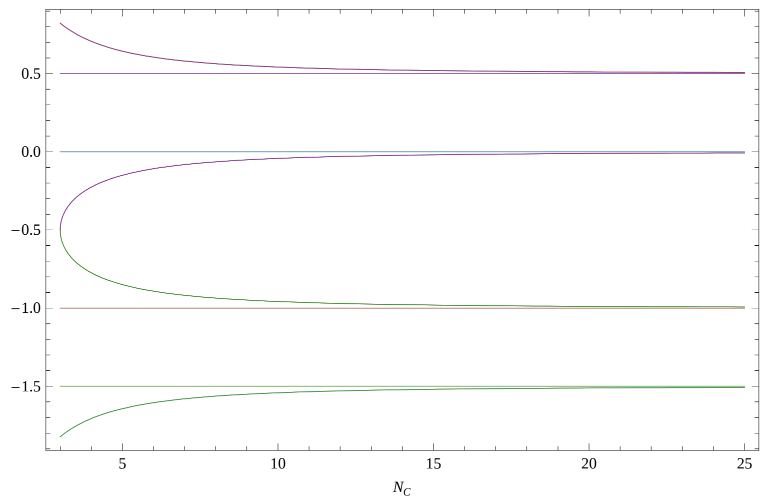

Figure 1: N_c dependence of the eigenvalues of the model $\mathrm{TOY}_{\mathrm{Adj,Fund}}$.

for $N_c > 3$ it is given by a 12×12 matrix. The leading eigenvalue as function of N_c can be easily computed

$$E_0(N_c) = \frac{\sqrt{10N_c^2 + 36 + 6\sqrt{N_c^4 + 36N_c^2 + 36}} - 2N_c}{4N_c} \tag{4.31}$$

and indeed goes to the value 1/2 in the large N_c limit. Let us note that if one considers the planar approximation (in the 2CT configuration), the leading eigenvalue would be underestimated with a relative error of $(E_0(3) - E_0(\infty))/E_0(3) \simeq 40\%$ w.r.t. the case $N_c = 3$. In Fig. 1 we report the N_c dependence of all the eigenvalues in the range $3 \leq N_c \leq 25$.

Similar models, but in a higher spin representation, can be constructed in order to maintain the integrability in the large N_c limit. One simply needs to consider for any irreducible representation s of the particles the function f to be a corresponding specific polynomial as described in [22].

5 Conclusions

We have introduced a family of dynamical models describing interacting particles with color and spin degrees of freedom. The main motivation was to study within this framework how much the large N_c approximation is significant when one is trying to extract the spectrum of these quantum systems.

Indeed in some relevant physical cases the only results available are restricted to the case with a planar structure resulting from the large N_c approximation, when integrability arises and gives the possibility to exactly solve the problem. These facts are seen when considering

QCD scattering amplitudes in the Regge limit and LLA approximation, characterized by the BKP dynamics.

We have focused our study to the the case of four particles and considered in details three toy models. One toy model (case (c) in section 4.2) was considered to test our computational method based on group theory since one is able to make a direct comparison with results already known from other methods used in spectroscopy.

The first model presented in section 4.2 (a) is aimed to mimic to some extend the behavior of the 4 gluon BKP kernel, since we have forced it to have in the large N_c limit the same leading eigenvalues of the BKP system for both one and two cylinder topologies. We were able to compute the different eigenvalues of this toy model as function of N_c and we have found that the leading one at $N_c = 3$ present corrections of almost 30% w.r.t. the planar approximation, which one may understand in terms of a large coefficient in the $1/N_c^2$ correction term. The mixing in color-spin configuration structure has been also studied.

Another model (case (b) in section 4.2) was considered since in the large N_c limit it gives rise to the one cylinder topology Heisenberg XXX spin chain which is integrable. For the spin 1/2 case we have found at finite $N_c = 3$ corrections to the leading eigenvalue of about 40%.

Let us note that in our analysis we were restricting ourselves to study the toy model Hamiltonians on the space of states which are singlet with respect to the $SU(2)$ "spin" configurations. This was a choice dictated by technical reasons but one should look forward to extending the investigation to all the possible states.

These kind of models and possibly more general ones appear to be interesting also by themselves and we feel that they deserve more studies in order to see, for example, if some remnant from integrability can be traced back at finite N_c.

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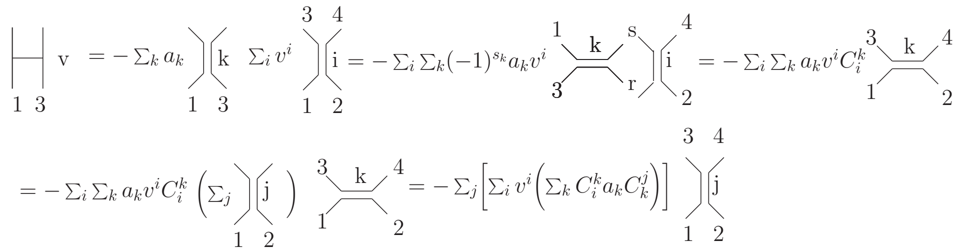
Appendix A

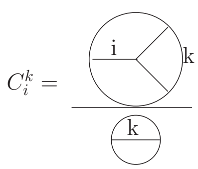
In this appendix we note a few facts about the crossing matrices introduced in section 3 and 4 for the $SU(N_c)$ group. Related considerations may be found in [24, 25, 26, 27, 28] where explicit expressions for the crossing matrices can be found and therefore will not be given here.

Let us rewrite in graphic notation the operator $\vec{T}_i\vec{T}_j$ in the basis (i, j) and the color vector state in the basis $(1, 2)$.

Let us compute the first non trivial crossing case, $\vec{T}_1 \vec{T}_3\, v$, remembering to rewrite the final result again in the basis $(1,2)$. In a graphical notation we have



where the crossing matrix (essentially 6j symbols) can be written as



In a similar way one can also trace the action of the $\vec{T}_1 \vec{T}_4$ operator. One can see that in the last relation there is an asymmetry due to the fact that one divides by the dimension of the k-representation. It is convenient to perform a similarity transformation to work with a symmetric crossing matrix. For this purpose it is sufficient to introduce the matrix $\Delta = \mathrm{diag}(d_i)$ and define the new symmetric matrix $C \to \Delta^{-\frac{1}{2}} C \Delta^{\frac{1}{2}}$ which acts on the vectors with components $v^i \to \left(\Delta^{-\frac{1}{2}} v\right)^i$.

Appendix B

Let us consider the BKP toy model described and analysed in section 4.1. From numerical investigations one finds that the leading eigenvector v_0 and the two closest subleading $v_{1,2}$ at $N_c = 3$ have the following components

$$
v_0 \simeq \begin{pmatrix} 0.590 & P_1 Q_1 \\ 0.085 & P_1 Q_5 \\ 0.344 & P_{8A} Q_3 \\ 0.199 & P_{8S} Q_1 \\ 0.199 & P_{8S} Q_5 \\ 0.293 & P_{10+\bar{10}} Q_3 \\ 0.179 & P_{27} Q_1 \\ 0.574 & P_{27} Q_5 \end{pmatrix}
\quad
v_1 \simeq \begin{pmatrix} 0.166 & P_1 Q_3 \\ 0.342 & P_{8A} Q_1 \\ 0.317 & P_{8A} Q_5 \\ 0.385 & P_{8S} Q_3 \\ 0.267 & P_{10+\bar{10}} Q_1 \\ 0.598 & P_{10+\bar{10}} Q_5 \\ 0.421 & P_{27} Q_3 \end{pmatrix}
\quad
v_2 \simeq \begin{pmatrix} -0.775 & P_1 Q_1 \\ 0.002 & P_1 Q_5 \\ 0.008 & P_{8A} Q_3 \\ 0.123 & P_{8S} Q_1 \\ 0.114 & P_{8S} Q_5 \\ 0.268 & P_{10+\bar{10}} Q_3 \\ 0.151 & P_{27} Q_1 \\ 0.525 & P_{27} Q_5 \end{pmatrix}
$$

As one can see the eigenvector v_0 of the highest eigenvalue is even which the two fold

degerate next lager eigenvalue has eigenstates of both parities (v_1 odd and v_2 even).

In the large N_c limit case the eigenvectors of the three fold degenerate leading eigenvalue of the 2 cylinder topology are

$$w_0^{(2CT)} \simeq \begin{pmatrix} 1 & P_1 Q_1 \end{pmatrix} \qquad w_1^{(2CT)} \simeq \begin{pmatrix} \frac{1}{3} & P_{10+\bar{10}} Q_1 \\ \frac{\sqrt{5}}{3} & P_{10+\bar{10}} Q_5 \\ \frac{1}{\sqrt{6}} & P_{27} Q_3 \\ \frac{1}{\sqrt{6}} & P_0 Q_3 \end{pmatrix} \ w_2^{(2CT)} \simeq \begin{pmatrix} \frac{1}{\sqrt{3}} & P_{10+\bar{10}} Q_3 \\ \frac{1}{3\sqrt{2}} & P_{27} Q_1 \\ \frac{\sqrt{5}}{3\sqrt{2}} & P_{27} Q_5 \\ \frac{1}{3\sqrt{2}} & P_0 Q_1 \\ \frac{\sqrt{5}}{3\sqrt{2}} & P_0 Q_5 \end{pmatrix}$$

Again also in this system we can track the same parity properties, which are invariant under the flow in N_c.

Similarly one may investigate the states associated to the one cylinder topology at $N_c = \infty$ and their corresponding partners at finite N_c. For brevity we just report here the two most relevant states in the N_c infinity limit:

$$w_0^{(1CT)} \simeq \begin{pmatrix} z_1 & P_{8A} Q_1 \\ z_3 & P_{8S} Q_3 \\ z_5 & P_{8A} Q_5 \end{pmatrix} \qquad w_1^{(1CT)} \simeq \begin{pmatrix} z_1 & P_{8S} Q_1 \\ z_3 & P_{8A} Q_3 \\ z_5 & P_{8S} Q_5 \end{pmatrix} \quad w_2^{(1CT)} \simeq \begin{pmatrix} 0.245 & (P_0 Q_1 - P_{27} Q_1) \\ 0.663 & (P_0 Q_5 - P_{27} Q_5) \end{pmatrix}$$

where $z_1 \simeq 0.815$, $z_3 \simeq 0.405$ and $z_5 \simeq 0.415$. We stress that $w_2^{(1CT)}$ has no corrispective at $N_c = 3$.

Appendix C

This appendix is devoted to check in one specific case that our approach gives result in agreement with other methods widely used in spectroscopy. We start by considering a system of n particles in the bifundamental representation of $SU(N_c) \times SU(2)$, characterized by an Hamiltonian (4.18) (with f the identity function)

$$\mathcal{H}_n = -\frac{1}{N_c} \sum_{1 \le k < l \le n} \vec{T}_k \vec{T}_l \ \ \vec{S}_k \vec{S}_l \, , \tag{5.32}$$

which can be written in terms of the quadratic Casimir operators of $SU(N_c)$, $SU(2)$ and $SU(2N_c) \supset SU(N_c) \times SU(2)$ (see [29]).

Indeed the tensor products of $\vec{T}_k \vec{S}_l$ are among the generators of $SU(2N_c)$, so it is useful to introduce the entire algebras for this group

$$\alpha_k = \begin{cases} \frac{1}{\sqrt{N_c}} S_l & k = 1, 2, 3 = l \\ \frac{1}{\sqrt{2s+1}} T_a & k = 4, \ldots, N_c^2 + 2; \ a = 1, \ldots, N_c^2 - 1 \\ \sqrt{2}\, T_k S_l & k = N^2 + 3, \ldots, 4N_c^2 - 1; \ l = 1, 2, 3 \end{cases} \tag{5.33}$$

$SU(2)\otimes SU(2)$	$SU(4)$	$(\mu_1,\mu_2,\mu_3)\equiv\mathcal{R}_SU(4)$
$\mathbf{1}\otimes\mathbf{1}$	$1,20,35$	$(0,0,0)(0,2,0)(4,0,0)$
$\mathbf{1}\otimes\mathbf{3}$	$15,45$	$(1,0,1)(2,1,0)$
$\mathbf{1}\otimes\mathbf{5}$	20	$(0,2,0)$
$\mathbf{3}\otimes\mathbf{3}$	$15,20,35,45$	$(1,0,1)(0,2,0)(4,0,0)(2,1,0)$
$\mathbf{3}\otimes\mathbf{5}$	45	$(2,1,0)$

Table 1: Correspondence between *irreps* of $SU(4)$ and $SU(2)\otimes SU(2)$

with the normalization $Tr(\alpha_k\alpha_{k'}) = 1/2\,\delta_{kk'}$. The Hamiltonian for this system can be rewritten as

$$\mathcal{H}_{All-fund} = -\frac{1}{4N_c}\left[\mathcal{C}_{2N_c} - \frac{1}{N_c}\mathcal{C}_{N_c} - \frac{1}{2s+1}\mathcal{C}_2 - 2n\,\frac{N_c^2-1}{2N_c}s(s+1)\right], \tag{5.34}$$

where the quadratic Casimir operators $\mathcal{C}_n$ are defined as in [29] and $s = 1/2$. Note that all the operators introduced above depend on the irreducible representation of the symmetry group to which they refer to.

We are interested in the real representations so we set $N_c = 2$ and consider the case of only four particles. The symmetry group of the model becomes $SU(2)\otimes SU(2)\subset SU(4)$ and eq. (5.34), written for the four particle in a global singlet state, takes the form

$$\mathcal{H}_{All-fund} = -\frac{1}{8}\left[\mathcal{C}_4(\mathcal{R}) - \frac{9}{2}\right]. \tag{5.35}$$

In order to find its spectrum the next step consists of analyzing the irreducible representation content of each symmetry group of the model. So, for four particle with spin 1/2, one has (we specify also the multiplicity)

$$\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2} = 2(\mathbf{1}) + 3(\mathbf{3}) + \mathbf{5}, \tag{5.36}$$

and in the $SU(4)$ case

$$\mathbf{4}\otimes\mathbf{4}\otimes\mathbf{4}\otimes\mathbf{4} = \mathbf{1} + 3(\mathbf{15}) + 2(\mathbf{20}) + \mathbf{35} + 3(\mathbf{45}). \tag{5.37}$$

Then we need to study the $SU(2)\otimes SU(2)$ content of these $SU(4)$ *irrep*. This can be done using the results of [30] and in particular the entries of table 1, where the values in the third column are Dynkin indices.

So for particles in a total singlet state ($\mathbf{1}\otimes\mathbf{1}$) the Hamiltonian in eq. (5.35) admits four eigenvalues, each for a different *irrep* of $SU(4)$, with a 2-fold degenerate eigenvalue corresponded to $\mathbf{20_{SU(4))}}$ (see eq. (5.37)):

$$\begin{cases} -\frac{15}{16}, & \text{for } irrep\ \mathbf{35} \\ -\frac{3}{16}\ \ (2\text{x}), & \text{for } irrep\ \mathbf{20} \\ +\frac{9}{16}, & \text{for } irrep\ \mathbf{1} \end{cases} \tag{5.38}$$

and these are in perfect agreement with the spectrum evaluated with the method used previously throughout the paper (we take advantage from the formulas of [31] for the eigenvalues of a quadratic Casimir operator as functions of the Dynkin indices).

As a final remark we want to emphasize that the method of writing the Hamiltonian in terms of the Casimirs can be applied to systems with any number of particles (at the price, increasing their number, of a growing complexity in the induced irreducible representations) and moreover the analysis may not be restricted to singlet subspaces. Unfortunately it is not clear how to define a method for interacting particle not in the bifundamental representation.

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